ECUACIONES DIFERENCIALES
ECUACIONES DE VARIABLES SEPARABLES E100

Resolver las ecuaciones siguientes:

(1)
$$y' = 2x\sqrt{y-1}$$

(2)
$$y' = xy + x - 2y - 2; \qquad y(0) = 2$$

(3)
$$\frac{dT}{dt} = k(T - T_1),\ T(0) = T_0;\ k,\ T_0,\ T_1 \text{ constantes}$$

(4)
$$x^2 y' = 1 - x^2 + y^2 - x^2 y^2$$

(5)
$$(x^2+1)y' \tan y = x$$

(6)
$$xy' - y = 2x^2 y$$

(7)
$$\frac{dy}{dx} = \frac{(y-1)(x-2)(y+3)}{(x-1)(y-2)(x+3)}$$

(8)
$$\frac{dy}{dx} = \frac{\operatorname{sen} x + e^{2y}\operatorname{sen} x}{3e^y + e^y \cos 2x}; \qquad y\left(\frac{\pi}{2}\right) = 0$$

(9)
$$x^3 e^{2x^2+3y^2}\,dx - y^3 e^{-x^2-2y^2}\,dy = 0$$

(10)
$$\frac{dy}{dx} = \frac{y+1}{\sqrt{x} + \sqrt{xy}}$$

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Respuestas

(1)

$$y' = 2x\sqrt{y-1}$$

$$\begin{aligned}
\frac{dy}{dx} &= 2x(y-1)^{\frac{1}{2}} \Rightarrow \\
(y-1)^{-\frac{1}{2}}\,dy &= 2x\,dx \\
\int (y-1)^{-\frac{1}{2}}\,dy &= 2\int x\,dx \\
2(y-1)^{\frac{1}{2}} &= x^2 + c \qquad , c \text{ constante}
\end{aligned}$$

Elevando al cuadrado

$$\begin{aligned}
4(y-1) &= (x^2+c)^2 \\
y &= 1 + \frac{1}{4}(x^2+c)^2
\end{aligned}$$

(2)

$$y' = xy + x - 2y - 2; \qquad y(0) = 2$$

$$\begin{aligned}
\frac{dy}{dx} &= x(y+1) - 2(y+1) \\
&= (y+1)(x-2) \\
\frac{dy}{y+1} &= (x-2)\,dx \Rightarrow \\
\int \frac{dy}{y+1} &= \int (x-2)\,dx \Rightarrow \\
\ln(y+1) &= \frac{1}{2}(x-2)^2 + c; \qquad c \text{ constante}
\end{aligned}$$

Considerando la condición inicial $y(0) = 2$

$$\ln 3 = \frac{1}{2}(-2)^2 + c \Rightarrow c = \ln 3 - 2$$

Por lo que

$$\ln(y+1) = \frac{1}{2}(x-2)^2 + \ln 3 - 2$$

De donde

$$\begin{aligned} y+1 &= e^{\frac{1}{2}(x-2)^2+\ln 3-2} \\ &= e^{\frac{1}{2}(x-2)^2}e^{\ln 3}e^{-2} \\ y &= 3e^{\frac{1}{2}(x-2)^2}e^{-2}-1 \Rightarrow \\ y &= 3e^{\frac{1}{2}(x-2)^2-2}-1 \end{aligned}$$

(3)

$$\frac{dT}{dt} = k(T-T_1),\ T(0)=T_0;\ k,\ T_0,\ T_1 \text{ constantes}$$

$$\begin{aligned} \frac{dT}{T-T_1} &= k\,dt \Rightarrow \\ \int \frac{dT}{T-T_1} &= \int k\,dt \\ \ln(T-T_1) &= kt+c_1 \\ T-T_1 &= e^{kt+c_1} \\ &= e^{kt}e^{c_1} = ce^{kt} \qquad \text{con } c=e^{c_1} \\ T &= T_1+ce^{kt} \end{aligned}$$

Considerando la condición inicial $T(0)=T_0$

$$T_0 = T_1 + ce^0 \Rightarrow c = T_0 - T_1$$

Por lo tanto

$$T(t) = T_1 + (T_0-T_1)e^{kt}$$

(4)

$$x^2y' = 1 - x^2 + y^2 - x^2y^2$$

$$\begin{aligned}
x^2\frac{dy}{dx} &= (1-x^2)+y^2(1-x^2)\\
x^2\frac{dy}{dx} &= (1-x^2)(1+y^2)\\
\frac{dy}{1+y^2} &= \frac{1-x^2}{x^2}\,dx \Rightarrow\\
\int\frac{dy}{1+y^2} &= \int(x^{-2}-1)\,dx\\
\arctan y &= -\frac{1}{x}-x+c=\frac{-1-x^2+cx}{x}\Rightarrow\\
y &= \tan\left(\frac{-x^2+cx-1}{x}\right)
\end{aligned}$$

(5)

$$(x^2+1)y'\tan y = x$$

$$\begin{aligned}
(x^2+1)\frac{dy}{dx}\tan y &= x \Rightarrow\\
\tan y\,dy &= \frac{x\,dx}{x^2+1}\\
\int\frac{\operatorname{sen} y}{\cos y}\,dy &= \int\frac{x\,dx}{x^2+1}\\
-\ln(\cos y) &= \frac{1}{2}\ln(x^2+1)+c_1 \qquad c_1 \text{ constante.}\\
\ln(\cos y)^{-1} &= \ln(x^2+1)^{\frac{1}{2}}+c_1\\
(\cos y)^{-1} &= e^{\ln(x^2+1)^{\frac{1}{2}}+c_1}\\
&= e^{\ln(x^2+1)^{\frac{1}{2}}}e^{c_1}\\
\frac{1}{\cos y} &= c(x^2+1)^{\frac{1}{2}}\Rightarrow\\
\sec y &= c\sqrt{x^2+1}\Rightarrow\\
y &= \operatorname{arcsen}(c\sqrt{x^2+1})
\end{aligned}$$

(6)

$$xy'-y=2x^2y$$

$$
\begin{aligned}
x\frac{dy}{dx} &= 2x^2y + y = (2x^2+1)y \\
\frac{dy}{y} &= \frac{2x^2+1}{x}\,dx \Rightarrow \\
\int \frac{dy}{y} &= \int \left(2x + \frac{1}{x}\right) dx \Rightarrow \\
\ln y &= x^2 + \ln x + c_1 \qquad c \text{ constante} \Rightarrow \\
y &= e^{x^2+\ln x + c_1} \\
&= e^{x^2} e^{\ln x} e^{c_1} \qquad \text{pero } e^{c_1} = c \text{ constante} \Rightarrow \\
y &= e^{x^2} x c \Rightarrow \\
y &= cxe^{x^2}
\end{aligned}
$$

(7)

$$\frac{dy}{dx} = \frac{(y-1)(x-2)(y+3)}{(x-1)(y-2)(x+3)}$$

$$
\begin{aligned}
\frac{y-2}{(y-1)(y+3)}\,dy &= \frac{x-2}{(x-1)(x+3)}\,dx \\
\int \frac{y-2}{(y-1)(y+3)}\,dy &= \int \frac{x-2}{(x-1)(x+3)}\,dx
\end{aligned}
$$

Integrando mediante fracciones parciales

$$-\frac{1}{4}\int \frac{dy}{y-1} + \frac{5}{4}\int \frac{dy}{y+3} = -\frac{1}{4}\int \frac{dx}{x-1} + \frac{5}{4}\int \frac{dx}{x+3}$$

Multiplicando por 4, e integrando

$$
\begin{aligned}
-\ln(y-1) + 5\ln(y+3) &= -\ln(x-1) + 5\ln(x+3) + c_1 \\
\ln(y+3)^5 - \ln(y-1) &= \ln(x+3)^5 - \ln(x-1) + \ln c \\
\ln\left[\frac{(y+3)^5}{y-1}\right] &= \ln\frac{c(x+3)^5}{x-1} \Rightarrow \\
\frac{(y+3)^5}{y-1} &= \frac{c(x+3)^5}{x-1} \Rightarrow \\
(y+3)^5(x-1) &= c(x+3)^5(y-1)
\end{aligned}
$$

(8)

$$\frac{dy}{dx} = \frac{\operatorname{sen} x + e^{2y}\operatorname{sen} x}{3e^{y} + e^{y}\cos 2x}; \qquad y\left(\frac{\pi}{2}\right) = 0$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(\operatorname{sen} x)(1+e^{2y})}{e^{y}(3+\cos 2x)} \Rightarrow \\
\frac{e^{y}}{1+e^{2y}}\,dy &= \frac{\operatorname{sen} x}{3+\cos 2x}\,dx \Rightarrow \\
\int \frac{e^{y}dy}{1+e^{2y}} &= \int \frac{\operatorname{sen} x}{3+\cos 2x}\,dx
\end{aligned}$$

Pero $\cos^2 x = \frac{1}{2}(1+\cos 2x) \Rightarrow$

$$\begin{aligned}
\int \frac{e^{y}dy}{1+(e^{y})^2} &= \int \frac{\operatorname{sen} x\,dx}{3+2\cos^2 x - 1} \\
&= \int \frac{\operatorname{sen} x\,dx}{2+2\cos^2 x} \Rightarrow \\
\arctan e^{y} &= \frac{1}{2}\int \frac{\operatorname{sen} x\,dx}{1+(\cos x)^2} \Rightarrow \\
\arctan e^{y} &= \frac{1}{2}\arctan(\cos x) + c
\end{aligned}$$

Considerando la condición inicial $y\left(\frac{\pi}{2}\right) = 0$

$$\begin{aligned}
\arctan e^{0} &= \frac{1}{2}\arctan\left(\cos\frac{\pi}{2}\right) + c \\
\arctan 1 &= \frac{1}{2}\arctan 0 + c \Rightarrow c = \frac{\pi}{4}
\end{aligned}$$

Por lo tanto, la solución buscada es

$$\arctan e^{y} = \frac{1}{2}\arctan(\cos x) + \frac{\pi}{4}$$

Es decir

$$2\arctan e^{y} - \arctan(\cos x) - \frac{\pi}{2} = 0$$

(9)

$$x^3 e^{2x^2+3y^2}\,dx - y^3 e^{-x^2-2y^2}\,dy = 0$$

$$\begin{aligned} x^3e^{2x^2}e^{3y^2}\,dx &= y^3e^{-x^2}e^{-2y^2}\,dy \\ x^3e^{2x^2}e^{x^2}\,dx &= y^3e^{-2y^2}e^{-3y^2}\,dy \Rightarrow \\ \int x^2e^{3x^2}\,xdx &= \int y^2e^{-5y^2}\,y\,dy \end{aligned}$$

Integrando por partes a ambas integrales

$$\begin{aligned} \frac{1}{6}x^2e^{3x^2} - \frac{1}{3}\int e^{3x^2}\,xdx &= -\frac{1}{10}y^2e^{-5y^2} + \frac{1}{5}\int e^{-5y^2}\,y\,dy \\ \frac{1}{6}x^2e^{3x^2} - \frac{1}{8}e^{3x^2} &= -\frac{1}{10}y^2e^{-5y^2} - \frac{1}{50}e^{-5y^2} + c_1 \end{aligned}$$

Multiplicando por 450

$$(75x^2 - 25)e^{3x^2} + (45y^2 - 9)e^{-5y^2} = c$$

(10)

$$\frac{dy}{dx} = \frac{y+1}{\sqrt{x}+\sqrt{xy}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y+1}{\sqrt{x}+\sqrt{x}\,\sqrt{y}} \\ &= \frac{y+1}{\sqrt{x}(1+y)} \\ \frac{1+\sqrt{y}}{y+1}\,dy &= \frac{dx}{\sqrt{x}} \Rightarrow \\ \int \frac{\sqrt{y}+1}{y+1}\,dy &= \int x^{-\frac{1}{2}}\,dx \end{aligned}$$

Resolviendo la primera integral mediante el cambio de variable $\sqrt{y} = t$ se obtiene

$$\begin{aligned} 2\sqrt{y} + \ln(y+1) - 2\arctan\sqrt{y} &= 2\sqrt{x} + c \\ 2(\sqrt{y} - \sqrt{x}) + \ln(y+1) - 2\arctan\sqrt{y} &= c \end{aligned}$$